

Continuous variable steering and incompatibility via state-channel duality

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The term Einstein-Podolsky-Rosen steering refers to a quantum correlation intermediate between entanglement and Bell nonlocality, which has been connected to another fundamental quantum property: measurement incompatibility. In the finite-dimensional case, efficient computational methods to quantify steerability have been developed. In the infinite-dimensional case, however, less theoretical tools are available. Here, we approach the problem of steerability in the continuous variable case via a notion of state-channel correspondence, which generalizes the well-known Choi-Jamiołkowski correspondence. Via our approach we are able to generalize the connection between steering and incompatibility to the continuous variable case and to connect the steerability of a state with the incompatibility breaking property of a quantum channel, e.g., noisy NOON states and amplitude damping channels. Moreover, we apply our methods to the Gaussian steering setting, proving, among other things, that canonical quadratures are sufficient for steering Gaussian states.

Introduction.—The phenomenon of Einstein-Podolsky-Rosen (EPR) steering combines two central features of quantum theory: entanglement and incompatibility, namely, the impossibility of determine precisely and simultaneously certain properties of a physical system, e.g., position and momentum. In practice, steering is a quantum effect by which one experimenter, Alice, can remotely prepare (i.e., steer) an ensemble of states for another experimenter, Bob, by performing local measurement on her half of a bipartite system shared by them, and communicating the results to Bob [1].

Due to the fact that steering is a form of quantum correlation intermediate between entanglement and Bell nonlocality [2], it has been proven useful to solve foundational problems [3–7] and important for applications in quantum information processing such as one-sided-device-independent (1SDI) quantum information [8–10].

In the finite-dimensional case, several methods are available to attack the steering problem. In particular, efficient methods based on semidefinite programming [11] are able to detect and quantify steerability of a given state and set of measurements [3, 12–14]. Notwithstanding the existence of several methods, see, e.g., [1, 15–19] and the review [14], such a systematic approach is missing in the continuous variable case.

In this paper, we will develop a general tool for discussing steering in the continuous variable case, which is based on an extension of the Choi-Jamiołkowski state-channel duality [20–22]. The Choi-Jamiołkowski correspondence associates a state to each channel, but not all states can be mapped to a channel in this way. We will extend this idea by showing that one can associate to each bipartite state a channel, such that the steerability

property of a state is equivalent to the property of the corresponding channel being incompatibility breaking [23], when all possible measurements are allowed for steering. This result, in turn, extends to the continuous variable case the result on equivalence between steering and joint-measurability [24–26].

In addition to these conceptual results, we find that the channel picture reduces seemingly different steering problems to a single one. For instance, we show that steerability of noisy NOON-states (cf. Ref. [19]) corresponds to the decoherence of incompatibility under an amplitude damping channel (cf. [27, 28]), and how to use steering to investigate its Markovianity properties. Using incompatibility techniques we investigate both analytically and numerically the noise tolerance of these states with two quadrature measurements. Finally, we apply our methods in the continuous variable Gaussian settings, showing that steerability by a pair of canonical quadrature measurements already ensures steerability by all Gaussian measurements, and connecting this to Gaussian incompatibility breaking channels [29]. We also show in passing how the channel picture yields an independent proof of the known Gaussian steering criterion [1].

Preliminary notions.—The first notion we need is that of positive-operator-valued measure (POVM). For a discrete set of outcomes $\lambda \in \Lambda$, a POVM is a collection $\{M_\lambda\}_\lambda$ of positive semidefinite operators $M_\lambda \geq 0$ with $\sum_\lambda M_\lambda = \mathbb{1}$. Such operators represent the probability of the outcome λ for a measurement on a state ρ via the rule $\text{Prob}(\lambda) = \text{tr}[\rho M_\lambda]$. However, the above definition is not enough in the continuous-variable case, e.g., for position and momentum measurements where $\Lambda = \mathbb{R}^n$ with the usual integration measure $d\lambda$. A POVM $\{G_\lambda\}_{\lambda \in \Lambda}$, then, consists of elements G_λ which may be infinitesimal so that, in general, only the integrals $G(X) := \int_X G_\lambda d\lambda$ with $X \subset \Lambda$ define proper operators. This definition clarifies the name positive-operator-valued measure [30], i.e., a map from measurable sets to positive operators $X \mapsto \int_X G_\lambda d\lambda$ with normalisation $\int_\Lambda G_\lambda d\lambda = \mathbb{1}$ and ad-

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ditivity on disjoint sets. A typical example is the position operator (or the quadrature of an optical field) $Q = \int_{\mathbb{R}} q|q\rangle\langle q|dq$. The corresponding POVM has elements $\tilde{Q}_q = |q\rangle\langle q|$, which are not proper operators.

A collection of POVMs, with measurement settings labelled by x , will be denoted as $\mathcal{M} = \{M_{a|x}\}_{a,x}$ and called a *measurement assemblage*. In the discrete case, a measurement assemblage is said to be *jointly measurable* [30] if there exist a POVM $\{G_\lambda\}_\lambda$ such that each POVM element $M_{a|x}$ can be obtained from G_λ via classical postprocessing, i.e., $M_{a|x} = \sum_\lambda D(a|x, \lambda)G_\lambda$ for all x, a , where $D(a|x, \lambda) \geq 0$ and $\sum_a D(a|x, \lambda) = 1$. Similarly, for the continuous variable case, one has joint measurability if for each $X \subset \mathcal{A}_x$, with \mathcal{A}_x the set of outcomes for the setting x ,

$$M_{X|x} := \int_X M_{a|x} da = \int_\Lambda D(X|x, \lambda)G_\lambda d\lambda, \quad (1)$$

where for each x , $D(\cdot|x, \cdot) : \mathcal{A}_x \times \Lambda \rightarrow [0, 1]$ is again a postprocessing, called *weak Markov kernel* in the general context [31].

Regarding joint measurability, an important notion for our purposes is that of *incompatibility breaking channel* [23], namely, a quantum channel Λ such that the assemblage $\{\Lambda^*(M_{a|x})\}_{x,a}$ is compatible for any assemblage $\{M_{a|x}\}_{x,a}$. Here Λ^* denotes the Heisenberg picture of the channel Λ . For instance, entanglement breaking channels [32] belong to this class.

Another main ingredient for our discussions is bipartite quantum steering. Alice can prepare an ensemble of states for Bob by performing a local measurement (x) on her half of the bipartite system and communicating the result (a) to Bob. Such a procedure is related to the measurement assemblage $\{A_{a|x}\}_{a,x}$ via the rule $\varrho(a|x) := \text{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]/P(a|x)$, where $\varrho(a|x)$ is the reduced state obtained by Bob as a consequence of the measurement and $P(a|x) := \text{tr}[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]$ is the probability of the outcome a for the setting x .

We will call a collection of conditional (unnormalized) states $\{\rho_{a|x}\}_{a,x}$, with $\rho_{a|x} := P(a|x)\varrho(a|x) = \text{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]$, a *state assemblage*. They satisfy the nonsignalling rule $\rho_B = \sum_a \rho_{a|x}$ for all x , with $\rho_B := \text{tr}_A[\rho_{AB}]$ being the reduced state for Bob.

An assemblage $\{\rho_{a|x}\}_{a,x}$ is called *unsteerable* if it admits a local hidden state (LHS) model [1], i.e., a collection of positive operators $\{\sigma_\lambda\}_\lambda$ such that $\text{tr}[\sum_\lambda \sigma_\lambda] = 1$ and $\rho_{a|x} = \sum_\lambda D(a|x, \lambda) \sigma_\lambda$ for all a, x , with $D(a|x, \lambda) \geq 0$ and $\sum_a D(a|x, \lambda) = 1$. If a LHS model exists, Bob can interpret each $\rho_{a|x}$ as coming from some preexisting states σ_λ , where only the classical probabilities are updated due to the information obtained by Alice from her measurement. Similarly to the case of joint measurability, we define unsteerable assemblages $\sigma_x(X) := \int_X \sigma_{a|x} da$ for all $X \subset \mathcal{A}_x$, as those admitting a LHS defined by

$$\int_X \sigma_{a|x} da = \int_\Lambda D(X|x, \lambda) \sigma_\lambda d\lambda, \quad (2)$$

where $D(\cdot|x, \cdot)$ is a weak Markov kernel for each x .

State-channel correspondence.—The key idea to attack the steering problem in the continuous variable case is based on the notion of state-channel duality. A familiar example of this is the Choi-Jamiołkowski (CJ) isomorphism, which maps every channel $\mathsf{T} : \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_A)$ into a density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$, as $\rho_{\mathsf{T}} = (\mathsf{T} \otimes \text{Id})(|\Omega_0\rangle\langle\Omega_0|)$, where $|\Omega_0\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ is the maximally entangled state on $\mathcal{H}_B \otimes \mathcal{H}_B$ and $\dim \mathcal{H}_B = d < \infty$. In other words, the CJ isomorphism is a one-to-one mapping between channels and bipartite states ρ_{AB} with completely mixed \mathcal{H}_B marginals, i.e., $\sigma = \text{tr}_A(\rho) = \mathbb{1}/d$. It has been used in connection with steering for the definition of channel steering [33] and the verification of the quantumness of a channel [34]. We can extend this notion as follows.

Lemma 1. *There is a 1-to-1 correspondence between bipartite states ρ sharing a full-rank marginal $\sigma = \text{tr}_A[\rho]$, and quantum channels T from Bob to Alice, such that*

$$\rho = (\mathsf{T} \otimes \text{Id})(|\Omega\rangle\langle\Omega|) \quad (3)$$

where $|\Omega\rangle := \sum_{n=1}^d \sqrt{s_n} |nn\rangle \in \mathcal{H}_B \otimes \mathcal{H}_B$ is defined as the purification of $\sigma = \sum_n s_n |n\rangle\langle n|$.

Sketch of the proof. Given a channel T , ρ is clearly a valid state and $\text{tr}_A[\rho] = \sigma$. Viceversa, given ρ with marginal σ , the action of T can be computed as

$$\sigma^{\frac{1}{2}} \mathsf{T}^*(A) \sigma^{\frac{1}{2}} = \text{tr}_A[\rho(A \otimes \mathbb{1})]^\mathsf{T}, \quad (4)$$

cf. Appendix A 2 for a detailed derivation of Eq. (4). For $d < \infty$, one can invert $\sigma^{\frac{1}{2}}$ and solve for $\mathsf{T}^*(A)$. For $d = \infty$, we cannot directly invert $\sigma^{\frac{1}{2}}$, since it will be an unbounded operator. However, one can still construct the Kraus operators $\{M_k\}_k$ for the channel T^* from the Kraus operators R_k of the map $\sigma^{\frac{1}{2}} \mathsf{T}^*(\cdot) \sigma^{\frac{1}{2}}$, obtained via Eq. (4). This is achieved by proving that the operator $R_k \sigma^{-\frac{1}{2}}$ can be extended to a bounded operator on \mathcal{H}_B (cf. Appendix A 2 for full details).■

With the above notion, we can reformulate the equivalence between steering of a state assemblage and incompatibility of a measurement assemblage [26] in full generality and from a quantitative perspective [35, 36]

Proposition 1. *The state assemblage $\{\sigma_x(X)\}_{X,x}$ defined by ρ and $\{A_x\}_x$ is steerable \Leftrightarrow the measurement assemblage $\{\mathsf{T}^*(A_x)\}_x$ is incompatible. Here $\mathsf{T} \leftrightarrow \rho$ via Lemma 1, with $\sigma = \text{tr}_A[\rho] = \sigma_x(\mathcal{A}_x)$.*

Moreover, this correspondence is also quantitative, in the sense that the incompatibility robustness (IR) [26] of the measurement assemblage $\{\mathsf{T}^(A_x)\}_x$ coincide with the consistent steering robustness (CSR) [36] of the state assemblage $\{\sigma_x(X)\}_{X,x}$.*

Following [26], we call $B_x := \mathsf{T}^*(A_x)$ the *steering-equivalent observables* for the state assemblage, thereby extending the notion to the infinite-dimensional case. By Lemma 1, T^* has the required continuity properties to

make B_x proper POVMs (cf. Appendix A 3). When measured on Alice's side, on the purification $|\Omega\rangle$ of Lemma 1, the observables B_x reproduce the state assemblage $\{\sigma_x(X)\}_{X,x}$. In this way, *any* steering problem can be reduced to an equivalent one where the state is pure with full Schmidt rank. This case is then characterised by the observables alone; see Prop. 3 below. See Appendix A 3, for full details.

Another fundamental corollary of Prop. 1 relates unsteerable states to incompatibility breaking channels:

Proposition 2. *The state ρ is unsteerable \Leftrightarrow the channel T^* , defined in Lemma 1, is incompatibility breaking.*

The above results allow us to immediately recognize problems that are equivalent from a steering perspective. A well-known steering-equivalence is the one obtained via local unitary $\rho \mapsto (\mathbb{1} \otimes U)\rho(\mathbb{1} \otimes U^*)$ on Bob's side. It transforms the channel as $T^*(\cdot) \mapsto U^*T^*(\cdot)U$, which indeed does not affect incompatibility. In contrast, a local unitary on Alice's side may affect steerability: an arbitrary local channel S^* on Alice's side results in the concatenation $T^* \circ S^*(\cdot)$; hence the channel picture is especially suited for noisy settings.

Separable and pure states.—Consider separable states $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$, which are of course not steerable. We easily find the steering channel to be $T^*(A) = \sum_i \text{tr}[\rho_A^i A] F_i$, where $F_i = p_i \sigma^{-\frac{1}{2}} (\rho_B^i)^T \sigma^{-\frac{1}{2}}$ satisfies $0 \leq F_i \leq \mathbb{1}$ and $\sum_i F_i = \mathbb{1}$. Hence, we observe that the channel T is entanglement breaking [32].

Given a pure state $\rho = |\Psi\rangle\langle\Psi|$ with full Schmidt rank, we define a Hilbert-Schmidt operator $R: \mathcal{H}_B \rightarrow \mathcal{H}_A$ with $\langle n|R|m\rangle = \langle nm|\Psi\rangle$, where the basis on Bob's side is fixed as in Lemma 1, and arbitrary on Alice's side. Since R and R^* have full rank, $U = R\sigma^{-\frac{1}{2}}$ is unitary, and $|\Psi\rangle = (U \otimes \mathbb{1})|\Omega\rangle$, i.e. the corresponding channel is unitary: $T^*(A) = U^*AU$. As an infinite-dimensional example, the channel for the two-mode coherent state $|z\rangle$ with $z = re^{i\theta}$ is the phase shift $T^*(A) = e^{i\theta a^\dagger a} A e^{-i\theta a^\dagger a}$ if we identify the photon number bases of Alice and Bob. We obtain in this way the infinite-dimensional version of the result in Refs. [24, 25], namely

Proposition 3. *Given a full Schmidt rank $|\Psi\rangle$, its steerability via the measurements $\{A_{X|x}\}_{X,x}$ is equivalent to the incompatibility of $\{A_{X|x}\}_{X,x}$.*

The proof is straightforward since $|\Psi\rangle$ gives $T^* = \text{Id}$, with the usual identification of bases, up to a unitary transformation. Importantly, the problem of non-unique regularisation of maximally entangled states in $d = \infty$ is circumvented in the channel protocol.

Noisy NOON-states.—Consider the “NOON-state” $|N00N\rangle = \frac{1}{\sqrt{2}}(|0N\rangle - e^{iN\alpha}|N0\rangle)$ shared by Alice and Bob [37], with $\{|n\rangle\}$ photon number basis of 1-mode electromagnetic field. Via random photon loss, the state becomes $\rho_\eta = \eta|N00N\rangle\langle N00N| + (1-\eta)|00\rangle\langle 00|$, which is unsteerable for $\eta = 0$ and steerable for $\eta = 1$. Hence, there must be a critical threshold η_c (which depends on

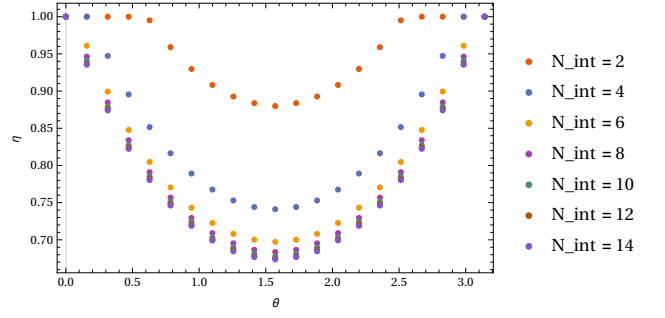


FIG. 1. Critical noise bound for steering through the 1001-state by a coarse-grained pair of quadrature measurements, as a function of the separation angle θ , with coarse-grainings of different number of intervals N_{int} . The case $N_{\text{int}} = 2$ can be reproduced analytically. Below $\eta = 2/3$ the setting is unsteerable by joint measurability criterion (7).

the allowed measurements) such that the state is steerable iff $\eta > \eta_c$ (cf. [19] and the references therein, for previous results on the problem). Using Lemma 1, we obtain the channel T corresponding to ρ_η as

$$T^*(A) = U^* \Lambda_r^*(A) U, \quad (5)$$

where $r = \sqrt{\eta/(2-\eta)}$, U is a unitary (irrelevant for steering), and $\Lambda_r(\cdot) = \sum_{i=0}^1 K_{i,r}(\cdot) K_{i,r}^*$ is the *amplitude damping channel* [38] defined for each $0 < r < 1$ by

$$K_{0,r} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}, \quad K_{1,r} = \begin{pmatrix} 0 & \sqrt{1-r^2} \\ 0 & 0 \end{pmatrix}; \quad (6)$$

see Appendix B 2. The problem, then, reduces to the question of how Λ_r breaks incompatibility. Before answering this, we shortly discuss how this problem arises in a seemingly different scenario, and how our techniques provide a solution in that case as well.

Consider a setup where physical noise arises on Alice's side due to coupling to a zero temperature heat bath. Starting from the 1001-state, the photon dissipates into the bath on Alice's side according to a channel \mathcal{E}_t given by the amplitude damping master equation [39] $d\mathcal{E}_t(\rho_0)/dt = \gamma(t) [\sigma_- \mathcal{E}_t(\rho_0) \sigma_+ - \frac{1}{2} \{\sigma_+ \sigma_-, \mathcal{E}_t(\rho_0)\}]$ where $\sigma_+ = |1\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$, and $\gamma(t) = -2\text{Re} \frac{d}{dt} \log G(t)$ with $G(t)$ depending on the bath spectral density. The state at time t is $\rho_t = (\mathcal{E}_t \otimes \text{Id})(|1001\rangle\langle 1001|)$ so by (3), its steering channel $T = T_t$ equals \mathcal{E}_t up to a unitary. Using the form of \mathcal{E}_t [27] we find $T_t^*(A) = U^* \Lambda_{r(t)}^*(A) U$, as in Eq. (5), where now $r(t) = |G(t)|$. Interestingly, in this scenario our state-channel duality connects the steerability problem with the non-Markovian properties of the bath (cf., e.g., [40]), previously associated with temporal correlations [41] and decoherence of incompatibility [27].

Having the two different scenarios mapped into one steering problem (for each fixed time t in the heat bath case), we introduce the following criterion for n “damped

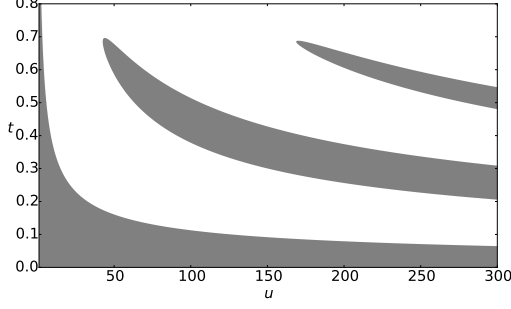


FIG. 2. Steerability region for the dynamical setting (shaded area). The parameter u is the coupling strength and t is time. The two revival regions reflect the non-Markovian character of the evolution in the strong coupling regime, which allows steerability to re-emerge at later times.

measurements" $\Lambda_r^*(A_x)$, $x = 1, \dots, n$ to be jointly measurable (cf. Appendix B 1 for a proof):

$$\sum_{x=1}^n \det \frac{\Lambda_r^*(A_{X|x})}{\langle 0|A_{X|x}|0 \rangle} \geq n-1 \text{ for each } X \subset \mathcal{A}_x. \quad (7)$$

We focus on the case of Alice attempting to steer Bob using rotated quadratures $Q_\theta = (e^{i\theta}a^\dagger + e^{-i\theta}a)/\sqrt{2}$. They act in the infinite-dimensional Hilbert space, with spectral projections (PVM) $Q_{q|\theta} = e^{i\theta a^\dagger a}|q\rangle\langle q|e^{-i\theta a^\dagger a}$. As our state lives in $\text{span}\{|0\rangle, |N\rangle\}$, only the 2×2 -matrix $(\tilde{Q}_{q|\theta})_{nm} = \langle n|Q_{q|\theta}|m\rangle = e^{i\theta(n-m)}\langle n|q\rangle\langle q|m\rangle$ with $n, m = 0, N$ contributes. We assume that Alice only has one pair, i.e. an assemblage $\{\tilde{Q}_{q|0}, \tilde{Q}_{q|\theta}\}_q$ for fixed θ ; the criterion in Eq. (7) then reads $r^2 \leq 1/2$, independently of q , N , and θ . Hence, for steering we require $r_c \geq 1/\sqrt{2}$, i.e., $\eta_c \geq 2/3$. The result $\eta_c \approx 2/3$ (up to the SDP precision) was found for $N = 1$ in Ref. [19].

Independently of Ref. [19], we show that our method can provide also upper bounds on r_c for $N = 1$. First, by binarizing $\tilde{Q}_{q|\theta}$ ($q > 0$ or $q < 0$) and applying the analytical method in Ref. [42] we get $\eta_c \leq 2\pi/(4 + \pi) \approx 0.88$ for $\theta = \pi/2$. With a finer coarse graining, i.e., by dividing the real line in $N_{\text{int}} = 2, 4, 6, 8, 10, 12, 14$ parts, we get better upper bounds via SDP methods, cf. Fig. 1. With $N_{\text{int}} = 20$ and $\theta = \pi/2$, we obtain the value $\eta_c \leq 0.671$, which is rather close to the lower bound $\eta_c \geq 2/3$. See Appendix B 3 for details on numerical methods.

The result can now be applied to characterise steering in the heat bath scenario: for any time t , the state ρ_t is steerable by $\{Q_{q|0}, Q_{q|\pi/2}\}$ iff $r(t) \geq r_c$. For the typical Lorentzian spectral density, $r(t) = e^{-t/2}|\cosh(wt/2) + \sinh(wt/2)/w|$ where t is in units of inverse linewidth, and $w = \sqrt{1 - 2u}$ with u the coupling strength [27]. We can then evaluate $r(t) \geq r_c$ with the numerical value $r_c \approx 1/\sqrt{2}$, to get the region of points (u, t) where the state is steerable; cf. Fig. 2 and caption.

Gaussian channels.—We will now apply the ideas of Lemma 1 to the problem of Gaussian steering. Suppose

Alice and Bob both have a continuous variable system with N degrees of freedom. Gaussian states ρ correspond (up to displacements) to *covariance matrices* (CM) \mathbf{V} which are real symmetric and satisfy the uncertainty relation $\mathbf{V} + i\mathbf{\Omega} \geq 0$ [43], with $\mathbf{\Omega} = \bigoplus_{k=1}^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Gaussian channels are defined as those channels mapping Gaussian states into Gaussian states, and are thus characterized by their action on CMs. More precisely, a Gaussian channel corresponds (again up to shifts) to a pair of real matrices (\mathbf{M}, \mathbf{N}) that act on \mathbf{V} as $\mathbf{V} \mapsto (\mathbf{M}^T \mathbf{V} \mathbf{M} + \mathbf{N})$ and satisfy the complete positivity condition $\mathbf{C}_{\mathbf{M}, \mathbf{N}} + i\mathbf{\Omega} \geq 0$ where $\mathbf{C}_{\mathbf{M}, \mathbf{N}} = \mathbf{N} - i\mathbf{M}^T \mathbf{\Omega} \mathbf{M}$ [44]. We, then, have

Lemma 2. *There is a 1-to-1 correspondence between bipartite Gaussian states ρ sharing a marginal $\sigma = \text{tr}_A[\rho]$ with CM \mathbf{V}_σ of full symplectic rank, and Gaussian channels \mathbf{T} from Bob to Alice, such that (3) holds with $|\Omega\rangle$ having CM $\mathbf{V}_\Omega = \begin{pmatrix} \mathbf{V}_\sigma & \mathbf{S}^T \mathbf{Z} \mathbf{S} \\ \mathbf{S}^T \mathbf{Z} \mathbf{S} & \mathbf{V}_\sigma \end{pmatrix}$. Here \mathbf{S} is a symplectic matrix diagonalising \mathbf{V}_σ , and $\mathbf{Z} = \bigoplus_{i=1}^N \sqrt{\nu_i^2 - 1} \sigma_z$, with ν_i the symplectic eigenvalues of \mathbf{V}_σ .*

Sketch of the proof. The correspondence between the CM \mathbf{V} of ρ and the parameters (\mathbf{M}, \mathbf{N}) of \mathbf{T} is given by

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_A & \mathbf{\Gamma}^T \\ \mathbf{\Gamma} & \mathbf{V}_\sigma \end{pmatrix} \leftrightarrow \begin{cases} \mathbf{M} = (\mathbf{S}^T \mathbf{Z} \mathbf{S})^{-1} \mathbf{\Gamma} \\ \mathbf{N} = \mathbf{V}_A - \mathbf{M}^T \mathbf{V}_\sigma \mathbf{M} \end{cases},$$

$$\mathbf{V} + i\mathbf{\Omega} \geq 0 \Leftrightarrow \mathbf{C}_{\mathbf{M}, \mathbf{N}} + i\mathbf{\Omega} \geq 0,$$

where the equivalence of the positivity conditions is via Schur complements; cf. Appendix C for details. ■

Gaussian measurements are those providing a Gaussian outcome distribution for every Gaussian state; they are (up to shifts in outcomes) parametrised by pairs of matrices (\mathbf{K}, \mathbf{L}) with $\mathbf{C}_{\mathbf{K}, \mathbf{L}} \geq 0$ [45]. Gaussian PVMs $Q_{a|\mathbf{x}}$ with $a \in \mathbb{R}$ are determined by $\mathbf{K} = \mathbf{x} \in \mathbb{R}^{2N}$, and correspond to *quadratures* $Q_{\mathbf{x}} = \mathbf{x}^T \mathbf{R} = \int a Q_{a|\mathbf{x}} da$ where $\mathbf{R} = (Q_1, P_1, \dots, Q_N, P_N)^T$, with $[Q_i, P_j] = i\delta_{ij} \mathbb{1}$, are the basis modes. A pair $(Q_{\mathbf{x}}, Q_{\mathbf{y}})$ with $[Q_{\mathbf{x}}, P_{\mathbf{y}}] = i\mathbb{1}$ is called *canonical*. Any Gaussian POVM M_a with $a \in \mathbb{R}$ is, up to a shift, a “noisy” quadrature: $M_a = M_{a|\mathbf{x}, \xi} := \frac{1}{\xi\sqrt{2\pi}} \int e^{-\frac{1}{2}(a-a')^2/\xi^2} Q_{a'|\mathbf{x}} da'$ [45]. Noise exceeding the uncertainty limit renders quadratures jointly measurable:

Lemma 3. *The noisy versions $M_{\mathbf{x}, \xi}$ and $M_{\mathbf{y}, \xi'}$ of two quadratures $Q_{\mathbf{x}}, Q_{\mathbf{y}}$ are jointly measurable if and only if*

$$\xi\xi' \geq \|[Q_{\mathbf{x}}, P_{\mathbf{y}}]\|/2,$$

in which case they have a Gaussian joint measurement.

This result generalises a known joint measurability criterion for position and momentum [46–48]; see Appendix C for a proof. Using it we finally prove

Proposition 4. *Let ρ be a bipartite Gaussian state with CM \mathbf{V} , and \mathbf{M}, \mathbf{N} the matrices of the channel \mathbf{T} given by Lemma 2. The following are equivalent:*
(i) ρ is steerable by the set of Gaussian measurements.

- (ii) ρ is steerable by some canonical pair of quadratures.
- (iii) $\mathbf{V} + i(\mathbf{0} \oplus \mathbf{\Omega})$ is not positive semidefinite.
- (iv) (\mathbf{M}, \mathbf{N}) do not define a valid Gaussian observable.

See Appendix C for proof. The equivalence between (i) and (iii) was originally proven in [1]. Here we use Lemma 3 to show that quadratures are enough [(ii)]; this comes closest to the original notion of steering of an EPR-state via position and momentum as discussed by Schrödinger [49]. Furthermore, a new interpretation emerges from (iv): the Gaussian POVM determined by the *channel parameters* (\mathbf{M}, \mathbf{N}) is exactly the joint observable for the assemblage $\{\mathbf{T}^*(A_{\mathbf{a}}) : A_{\mathbf{a}} \text{ Gaussian}\}$ that rules out steering in (i) by Prop. 1. For details see Appendix C 5, where we also show that this is consistent with the LHS of [1]. Finally, Prop. 4 yields a new result on channels: a Gaussian channel which maps each pair of quadratures into a jointly measurable pair, must be *Gaussian incompatibility breaking* in the sense of [29].

Conclusions.— Steering is a genuine quantum phenomenon, with important applications both in quantum information processing and foundations of quantum mechanics. Notwithstanding the growing interest for it in the past few years [14], limited results and tools are available in the continuous variable case. We introduced a state-channel correspondence that allows us to discuss the steering problem in the continuous framework. In particular, we are able to extend many of the results previously known only in the finite-dimensional case, such as the mathematical equivalence of steering and joint-measurability problems [26] and the equivalence of steering and joint-measurability for the case of full Schmidt rank states [24, 25]. Moreover, via state-channel duality we are able to connect steerability properties of noisy NOON states with Markovianity properties of the corresponding channel, and to provide an analytical lower bound to the steerability noise threshold for any N . Finally, we apply our methods to the Gaussian setting, introducing a new channel characterization of steerability and proving that canonical quadratures are enough for steering. An interesting future direction would be to extensively investigate the capability of the state-channel duality to provide steering, joint measurability, and incompatibility breaking criteria in the continuous variable case for states, observables, and channels, respectively.

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Appendix A: State-channel duality in view of steering

In this section, we first briefly review the basic notions involved in steering and joint measurability problems, with particular emphasis on the less known case of

continuous variable systems. We then provide details of the proofs of Lemma 1 and Prop. 1.

1. Hidden state models and measurements in terms of POVMs

We now review the fact that hidden state models and general quantum observables can both be described by POVMs. Since we are interested in the infinite-dimensional case with POVMs having continuous outcome sets, some technical considerations are unavoidable, and we discuss them briefly. These technicalities are not essential for understanding the main text, but they are needed to make the proofs mathematically sound.

A POVM with a discrete outcome set Λ is a collection $\{G_\lambda\}_{\lambda \in \Lambda}$ of positive semidefinite operators such that $\sum_{\lambda \in \Lambda} G_\lambda = \mathbb{1}$. This notion is not sufficient for this paper, since we also consider Gaussian measurements. A POVM with a *continuous outcome set* is one for which $\Lambda = \mathbb{R}^n$, i.e. the Euclidean space. The space comes with the usual integration measure $d\lambda$, and a POVM $\{G_\lambda\}_{\lambda \in \Lambda}$ consist of elements G_λ which may be “infinitesimal” so that, in general, only the integrals $\int_U G_\lambda d\lambda$ with $U \subset \Lambda$ define proper operators. In fact, a POVM G is *by definition* [30] no more than the *Positive Operator (Valued) Measure* $X \mapsto \int_X G_\lambda d\lambda$ with normalisation $\int_\Lambda G_\lambda d\lambda = \mathbb{1}$. In order to illustrate this well-known technical issue with a typical example relevant for the main text, consider the position operator $Q = \int_{\mathbb{R}} q|q\rangle\langle q|dq$. The corresponding POVM has elements $|q\rangle\langle q|$, which are *not* proper operators as they map wave functions ψ into improper states $\psi(q)|q\rangle$. The symbols $|q\rangle\langle q|$ only make up operators when integrated into $\int_{[a,b]} |q\rangle\langle q|dq$, which projects ψ into the wave function coinciding with $\psi(q)$ for $a \leq q \leq b$ and vanishing elsewhere.

The connection between hidden state models and POVMs is fairly obvious when $d < \infty$ and Λ is discrete. Suppose now we have a general family $\{\sigma_\lambda\}_{\lambda \in \Lambda}$ of positive operators on Bob’s side of a bipartite setting. Here Λ is the set of hidden variables, either discrete or continuous as above. *The crucial difference to POVMs is that each σ_λ is a proper trace class operator, i.e. not “infinitesimal” even in the continuous case.* The function $\lambda \mapsto \sigma_\lambda$ must satisfy the technical condition of measurability in the trace class norm, to allow the (Bochner) integrals $\int f(\lambda)\sigma_\lambda d\lambda$ to exist with finite trace for every measurable scalar function f on Λ . We also assume the normalisation $\sum_\lambda \sigma_\lambda = \sigma$ (discrete case) and $\int \sigma_\lambda d\lambda = \sigma$ (continuous case), where σ is again a fixed density operator. Then there exists a unique POVM G with outcomes in Λ , satisfying

$$\sigma^{\frac{1}{2}} G_\lambda \sigma^{\frac{1}{2}} = \sigma_\lambda. \quad (\text{A1})$$

This is clear in the finite-dimensional case with finite outcome set Λ — we just multiply with $\sigma^{-\frac{1}{2}}$ which preserves positivity, and normalisation translates into $\sum_\lambda G_\lambda = \mathbb{1}$.

For $d = \infty$ we need a technical density argument analogous to that used in the proof of Lemma 1 (see below). In the case of continuous outcome set, (A1) is again understood via the corresponding integrals.

Suppose then that we start with a POVM $\{G_\lambda\}_{\lambda \in \Lambda}$; the question is how to get the states σ_λ . If Λ is discrete, this is trivial: we define $\sigma_\lambda := \sigma^{\frac{1}{2}} G_\lambda \sigma^{\frac{1}{2}}$. However, the case of continuous outcome set Λ introduces a subtlety: we have to show that the possibly infinitesimal POVM elements G_λ yield trace class operators σ_λ . In general, this is nontrivial, and follows from the Radon-Nikodym property of the trace class (cf. p. 79 of [50]). In the relevant case of a position operator (and more generally a Gaussian POVM), this is easier to prove: $\sigma^{\frac{1}{2}}|q\rangle\langle q|\sigma^{\frac{1}{2}}$ maps ψ into $\langle q|\sigma^{\frac{1}{2}}\psi\rangle\sigma^{\frac{1}{2}}|q\rangle$, which is indeed a proper wave function since $\sigma^{\frac{1}{2}}|q\rangle = \sum_n \sqrt{s_n} \langle n|q\rangle |n\rangle$ has finite norm $\sum_n s_n |\langle n|q\rangle|^2 < \infty$ for all q due to $\sum_n s_n < \infty$, assuming the basis functions are continuous (which is the case for the number basis considered in the main text).

2. Proof of the state-channel duality (Lemma 1)

We first briefly recap the usual Choi-Jamiołkowski correspondence. It associates to each quantum channel T (from Bob to Alice) a bipartite state on $\mathcal{H}_A \otimes \mathcal{H}_B$ via

$$\rho_{\mathsf{T}} = (\mathsf{T} \otimes \text{Id})(|\Omega_0\rangle\langle\Omega_0|), \quad (\text{A2})$$

where $|\Omega_0\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ on $\mathcal{H}_B \otimes \mathcal{H}_B$ is the maximally entangled state in dimension d with the standard basis. This correspondence is one-to-one onto the set of bipartite states ρ with $\text{tr}_A[\rho] = \mathbb{1}/d$, i.e. every state with a maximally mixed reduction on Bob's side can be associated with a unique channel T such that the above equation holds. However, the correspondence cannot be applied to states ρ that do not have this property, and hence it is quite useless when mapping states to channels.

A simple modification fixes this problem: we replace $\mathbb{1}/d$ with an arbitrary *full rank* density matrix σ on Bob's side, take an eigenbasis $\{|n\rangle\}$ of σ (instead of the standard basis), and define the purification

$$|\Omega\rangle := \sum_n \sqrt{s_n} |nn\rangle \in \mathcal{H}_B \otimes \mathcal{H}_B, \quad (\text{A3})$$

where $s_n > 0$ are the corresponding eigenvalues of σ . Notice that $\langle\Omega|\Omega\rangle = \sum_n s_n = \text{tr}[\sigma] = 1$, i.e. $|\Omega\rangle$ is a proper pure state even in the infinite-dimensional case. Now given any channel T , the state

$$\rho = (\mathsf{T} \otimes \text{Id})(|\Omega\rangle\langle\Omega|) \quad (\text{A4})$$

clearly has the property $\text{tr}_A[\rho] = \sigma$, so we have managed to produce more general states than ones obtained by the Choi-Jamiołkowski correspondence. We now need to prove that the new correspondence is one-to-one *onto the*

set of states with $\text{tr}_A[\rho] = \sigma$. We first compute

$$\begin{aligned} \text{tr}[\rho(A \otimes B)] &= \langle\Omega|\mathsf{T}^*(A) \otimes B|\Omega\rangle \\ &= \sum_{nm} \sqrt{s_n s_m} \langle nn|\mathsf{T}^*(A) \otimes B|mm\rangle \\ &= \sum_{nm} \sqrt{s_n s_m} \langle n|\mathsf{T}^*(A)|m\rangle \langle n|B|m\rangle \\ &= \sum_{nm} \langle n|\sqrt{\sigma} \mathsf{T}^*(A) \sqrt{\sigma}|m\rangle \langle n|B|m\rangle \\ &= \text{tr}[\sqrt{\sigma} \mathsf{T}^*(A) \sqrt{\sigma} B^\top], \end{aligned} \quad (\text{A5})$$

where B^\top is the transpose of B in the fixed basis. Hence,

$$\sigma^{\frac{1}{2}} \mathsf{T}^*(A) \sigma^{\frac{1}{2}} = \text{tr}_A[\rho(A \otimes \mathbb{1})]^\top. \quad (\text{A6})$$

From this we see immediately that distinct channels correspond to distinct states, since the matrix elements of the channel are clearly uniquely determined by those of the channel: $\langle nm|\rho|n'm'\rangle = \text{tr}[\sqrt{\sigma} \mathsf{T}^*(|n'\rangle\langle n|) \sqrt{\sigma} (|m'\rangle\langle m|)^\top] = \sqrt{s_m} \sqrt{s_{m'}} \langle m'|\mathsf{T}^*(|n'\rangle\langle n|)|m\rangle$, where we have now also fixed a basis $\{|n\rangle\}$ on Alice's side.

What remains to be shown is that for *any* state ρ with $\text{tr}_A[\rho] = \sigma$ there exists a channel T such that (A4) (or, equivalently, (A6)) holds. If $d < \infty$ we can invert $\sigma^{-\frac{1}{2}}$ in (A6) to solve for $\mathsf{T}^*(A)$; however, we still need to show that this defines a channel, i.e. a CPTP map. We therefore proceed by writing the state ρ as

$$\rho = \sum_k |\psi_k\rangle\langle\psi_k| = \sum_{\substack{k,n,m \\ n',m'}} \langle nm|\psi_k\rangle\langle\psi_k|n'm'\rangle |nm\rangle\langle n'm'| \quad (\text{A7})$$

so that, for all bounded operators A and B (for Alice and Bob, respectively), we get

$$\text{tr}[\rho(A \otimes B)] = \sum_k \text{tr}[R_k^* A R_k B^\top], \quad (\text{A8})$$

where $R : \mathcal{H}_B \rightarrow \mathcal{H}_A$ is the Hilbert-Schmidt operator defined by $\langle n|R_k m\rangle = \langle nm|\psi_k\rangle$. Hence, $\text{tr}_A[\rho(A \otimes \mathbb{1})]^\top = \sum_k R_k^* A R_k$. In particular, $\sigma = \sigma^\top = \sum_k R_k^* R_k$.

Next, we need a little of functional analysis, so as to allow the proof to go through also for $d = \infty$, in which case the inverse of any full rank state is unbounded and requires some care. Let \mathcal{R} be the dense range of σ , containing all the basis vectors. Then $\mathcal{R} = \text{ran } \sigma^{\frac{1}{2}}$, $\sigma^{\frac{1}{2}}$ is injective, and for any $|\psi\rangle \in \mathcal{R}$ we have

$$\|R_k \sigma^{-\frac{1}{2}} \psi\|^2 \leq \sum_k \langle \sigma^{-\frac{1}{2}} \psi | R_k^* R_k \sigma^{-\frac{1}{2}} \psi \rangle = \|\psi\|^2,$$

which implies that each $R_k \sigma^{-\frac{1}{2}}$ extends to a bounded operator $M_k : \mathcal{H}_B \rightarrow \mathcal{H}_A$, for which $M_k \sigma^{\frac{1}{2}} = R_k$.

Since $\sum_k M_k^* M_k = \mathbb{1}$, the operators M_k set up a Kraus decomposition of a channel: we define

$$\mathsf{T}(T) := \sum_k M_k T M_k^* \quad (\text{A9})$$

for all (trace class) operators T . This is by construction completely positive, and it is trace-preserving since $\sum_k M_k^* M_k = \mathbb{1}$. In the infinite-dimensional case the series converges, e.g., in the weak topology. Plugging this channel in Eq. (A6) immediately gives

$$\begin{aligned} \sigma^{\frac{1}{2}} \mathsf{T}^*(A) \sigma^{\frac{1}{2}} &= \sum_k (M_k \sigma^{\frac{1}{2}})^* A M_k \sigma^{\frac{1}{2}} = \sum_k R_k^* A R_k \\ &= \text{tr}_A[\rho(A \otimes \mathbb{1})]^\top, \end{aligned} \quad (\text{A10})$$

so that (A6), and hence also (A4) holds, that is, the channel gives back the original state ρ . This proves that the correspondence is one-to-one, and completes the proof of Lemma 1 in the main text.

3. Proof of Proposition 1

For any fixed state σ , we have the correspondences

$$\begin{aligned} \{\mathsf{T}^*(A_{a|x})\} &\mapsto \{\rho_{a|x}\}, \\ \mathsf{T} &\mapsto \rho = (\mathsf{T} \otimes \text{Id})(|\Omega\rangle\langle\Omega|), \end{aligned} \quad (\text{A11})$$

between the measurement assemblage $A_{a|x}$ transformed, via the Heisenberg-picture channel T^* and the steering assemblage obtained via measurements $A_{a|x}$ on the state ρ . Note that the measurements $\{A_{a|x}\}$ stay fixed. Now, $\{\mathsf{T}^*(A_{a|x})\}$ is jointly measurable if and only if

$$\mathsf{T}^*(A_{a|x}) = \sum_\lambda D(a|x, \lambda) G_\lambda. \quad (\text{A12})$$

By multiplying this with $\sigma^{\frac{1}{2}}$ on both sides, we obtain

$$\rho_{a|x}^\top = \sum_\lambda D(a|x, \lambda) \sigma_\lambda, \quad (\text{A13})$$

where the hidden states σ_λ correspond to G_λ via (A1), and $\rho_{a|x}^\top := \sigma^{\frac{1}{2}} \mathsf{T}^*(A_{a|x}) \sigma^{\frac{1}{2}} = \text{tr}_A[\rho(A_{a|x} \otimes \mathbb{1})]^\top$ is the assemblage. As we have established above, all the correspondences are one-to-one, and hence steerability of the setting $(\rho, \{A_{a|x}\})$ is equivalent to the incompatibility of $\{\mathsf{T}^*(A_{a|x})\}$.

In order to prove the equivalence of the quantifiers, we recall from [36] that *consistent steering robustness* CSR are given by

$$\begin{aligned} \text{CSR}(\{\sigma_{a|x}\}) &= \inf \left\{ t \geq 0 \mid \{\pi_{a|x}\} \text{ } \sigma\text{-consistent}, \right. \\ &\quad \left. \left\{ \frac{\sigma_{a|x} + t\pi_{a|x}}{1+t} \right\} \text{ unsteerable} \right\}, \end{aligned} \quad (\text{A14})$$

where σ -consistence means $\sum_a \sigma_{a|x} = \sum_a \tau_{a|x}$ for all x . The incompatibility robustness, on the other hand, is given by

$$\begin{aligned} \text{IR}(\{M_{a|x}\}) &= \inf \left\{ t \geq 0 \mid \frac{M_{a|x} + tN_{a|x}}{1+t} \right. \\ &\quad \left. \text{jointly measurable} \right\}. \end{aligned} \quad (\text{A15})$$

We stress that these definitions, although typically interpreted as SDPs in the finite-dimensional case, can also be stated in infinite dimensions with possibly continuous outcomes for the measurements. We note that in such a case they can only be formulated as SDPs by first restricting to a subspace and discretising the outcomes, as in Appendix B 3 below.

Now, following a similar reasoning as the one in Ref. [35], we need to prove that for each noise term $N_{a|x}$ of the IR problem, i.e., a term making the measurement assemblage jointly measurable for a given t , we can find a noise term $\pi_{a|x}$ of the CSR problem, i.e., a term making the state assemblage unsteerable for the same t , and viceversa. We use again the relation

$$\pi_{a|x}^\top = \sigma^{\frac{1}{2}} N_{a|x} \sigma^{\frac{1}{2}} \quad (\text{A16})$$

to obtain a one-to-one mapping between σ -consistent assemblages and arbitrary POVMs. In the finite-dimensional case, we can argue as follows: Given $\{\pi_{a|x}\}_{a,x}$ a σ -consistent assemblage, $\{N_{a|x}\}_{a,x}$ defined as in Eq. (A16) is a valid measurement assemblage. Viceversa, given $\{N_{a|x}\}_{a,x}$ a valid measurement assemblage, we can construct the σ -consistent assemblage $\{\pi_{a|x}\}_{a,x}$ as

$$\pi_{a|x}^\top = \text{tr}_A [N_{a|x} \otimes \mathbb{1} |\Omega_\sigma\rangle\langle\Omega_\sigma|] = \sigma^{\frac{1}{2}} N_{a|x} \sigma^{\frac{1}{2}}, \quad (\text{A17})$$

where $|\Omega_\sigma\rangle := \sum_n \sqrt{s_n} |nn\rangle$ is the purification of $\sigma := \sum_n s_n |n\rangle\langle n|$. Hence $\text{CSR}(\{\sigma_{a|x}\}) = \text{IR}(\{\mathsf{T}^*(A_{a|x})\})$. When the Hilbert space is infinite-dimensional, with possibly continuous outcomes for the POVMs, we again need the argument in the last paragraph of Appendix A 1, since $N_{a|x}$ may not be a proper operator, while we need $\pi_{a|x}$ to actually be in the trace class. This establishes the correspondence (A16) between POVMs and σ -consistent assemblages in the same way as we obtained (A1). Then the equality $\text{CSR}(\{\sigma_{a|x}\}) = \text{IR}(\{\mathsf{T}^*(A_{a|x})\})$ clearly follows, and so we can extend the equivalence of quantifiers to the infinite-dimensional case.

The above reasoning also provides the connection with the *steering equivalent* observables defined in Ref. [26]. Given a state assemblage $\{\rho_{a|x}\}_{a,x}$, with reduced state $\sigma := \sum_a \rho_{a|x}$, we call *steering equivalent* (SE) observables the measurement assemblage $\{B_{a|x}\}_{a,x}$

$$B_{a|x} := \sigma^{-1/2} \rho_{a|x} \sigma^{-1/2}, \quad (\text{A18})$$

possibly restricting the expression to the range of σ , if σ is not full rank.

It is easy to show that if we have only access to the assemblage $\{\rho_{a|x}\}_{a,x}$, and not to the bipartite state ρ_{AB} , we can always interpret $B_{a|x}$ as the observables giving the assemblage when measured on the purification $|\Omega_\sigma\rangle := \sum_n \sqrt{s_n} |nn\rangle$ of $\sigma := \sum_n s_n |n\rangle\langle n|$. Namely,

$$\sigma^{1/2} B_{a|x} \sigma^{1/2} = \text{tr}_A [B_{a|x} \otimes \mathbb{1} |\Omega_\sigma\rangle\langle\Omega_\sigma|]^\top. \quad (\text{A19})$$

The main claim was that $\{\rho_{a|x}\}_{a,x}$ is unsteerable $\Leftrightarrow \{B_{a|x}\}_{a,x}$ is jointly measurable.

If we compare that with the definition of the channel T_ρ , we find that

$$\mathsf{T}_{|\Omega_\sigma\rangle\langle\Omega_\sigma|}^*(B_{a|x}) = B_{a|x}. \quad (\text{A20})$$

On the other hand, given a bipartite state ρ and a measurement assemblage $\{A_{a|x}\}_{a,x}$, we have that

$$\mathsf{T}_\rho^*(A_{a|x}) = B_{a|x}. \quad (\text{A21})$$

Then, Lemma 1 generalizes this interpretation to the case σ full rank reduced state on an infinite dimensional space.

Appendix B: Application to NOON-states

1. A joint measurability criterion for qubit POVMs with arbitrary outcome sets

We now prove the sufficient joint measurability criterion (7) of the main text. In contrast to most existing criteria, it applies to qubit POVMs with continuous outcome sets. In the main text, it was shown that this is useful for finding noise bounds for quadrature measurements restricted to two-dimensional photon number eigenspaces.

More generally, we prove that an assemblage of n qubit observables $\{B_{b|i}\}_{i=1}^n$ is jointly measurable if

$$\Delta(b_1, \dots, b_n) := \sum_i r_i(b_i) - n + 1 \geq 0, \quad (\text{B1})$$

where $r_i(b) := \det M_i(b)$, $M_i(b) := B_{b|i}/p_i(b)$, and $p_i(b) = \langle 0|B_{b|i}|0\rangle$. Indeed,

$$M_i(b) = \begin{pmatrix} 1 & \overline{f_i(b)} \\ f_i(b) & r_i(b) + |f_i(b)|^2 \end{pmatrix} \quad (\text{B2})$$

for some complex valued functions f_i . The normalisation forces $\int f_i(b)p_i(b)db = 0$ and $\int (|f_i(b)|^2 + r_i(b))p_i(b)db = 1$. We define G_{b_1, \dots, b_n} via

$$\frac{G_{b_1, \dots, b_n}}{\prod_{i=1}^n p_i(b_i)} = \begin{pmatrix} 1 & \sum_i \overline{f_i(b_i)} \\ \sum_i f_i(b_i) & |\sum_i f_i(b_i)|^2 + \Delta(b_1, \dots, b_n) \end{pmatrix}. \quad (\text{B3})$$

Using the constraints we see that it is normalised, and that

$$B_{b|i} = M_i(b)p_i(b) = \int \delta_{b,b_i} G_{b_1, \dots, b_n} db_1 \dots db_n. \quad (\text{B4})$$

The critical constraint is $G_{b_1, \dots, b_n} \geq 0$ now follows from

$$\det G_{b_1, \dots, b_n} = \Delta(b_1, \dots, b_n) \prod_{i=1}^n p_i(b_i)^2 \geq 0, \quad (\text{B5})$$

which is ensured by the assumption. This means that the B_i have a joint observable with deterministic response functions, so they are jointly measurable.

By taking $B_{a|i} = \Lambda_r^*(A_{a|i})$, where Λ_r is the amplitude damping channel defined in the main text, the assumption (B1) becomes (7), once we notice that $\langle 0|\Lambda_r^*(A_{a|i})|0\rangle = \langle 0|A_{a|i}|0\rangle$ (see (B7) below). Hence we have proved (7).

2. Analytical methods

Alice attempts to steer Bob using a pair of rotated quadratures Q_0, Q_θ , where $Q_\theta = (e^{i\theta}a^\dagger + e^{-i\theta}a)/\sqrt{2}$. This operator acts in the infinite-dimensional Hilbert space and its spectral projections (POVM) are given by $Q_{q|\theta} = e^{i\theta a^\dagger a}|q\rangle\langle q|e^{-i\theta a^\dagger a}$ with the continuum of “eigenvalues” q . However, our state $\rho = \eta|N00N\rangle\langle N00N| + (1-\eta)|00\rangle\langle 00|$ yields a channel

$$\begin{aligned} \mathsf{T}^*(A) &= \sigma^{-\frac{1}{2}} \text{tr}_A[\rho(A \otimes \mathbb{1})] \sigma^{-\frac{1}{2}} \\ &= \begin{pmatrix} r^2 A_{NN} + (1-r^2)A_{00} & -rA_{N0}e^{-iN\alpha} \\ -rA_{0N}e^{+iN\alpha} & A_{00} \end{pmatrix} \\ &= U^* \Lambda_r^*(A) U \end{aligned} \quad (\text{B6})$$

where $\sigma = \text{tr}_A(\rho) = (1-\eta/2)|0\rangle\langle 0| + \eta/2|N\rangle\langle N|$, $r = \sqrt{\eta/(2-\eta)}$,

$$\Lambda_r^*(A) = \sum_{i=0}^1 K_{i,r}^* A K_{i,r} = \begin{pmatrix} A_{00} & rA_{0N} \\ rA_{N0} & r^2 A_{NN} + (1-r^2)A_{00} \end{pmatrix} \quad (\text{B7})$$

is the amplitude damping channel (see, Eq. (6)) and

$$U := |0\rangle\langle N| - e^{iN\alpha}|N\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ -e^{iN\alpha} & 0 \end{pmatrix} \quad (\text{B8})$$

is a unitary matrix. Hence, the channel is supported in $\text{span}\{|0\rangle, |N\rangle\}$ and only the 2×2 -matrix $(\tilde{Q}_{q|\theta})_{nm} = \langle n|Q_{q|\theta}|m\rangle = e^{i\theta(n-m)}\langle n|q\rangle\langle q|m\rangle$ with $n, m = 0, N$ contributes. This gives, for the case $n, m = 0, N$

$$\tilde{Q}_{q|\theta} = \begin{pmatrix} 1 & e^{-iN\theta}h(q) \\ e^{iN\theta}h(q) & h(q)^2 \end{pmatrix} \frac{e^{-q^2}}{\sqrt{\pi}}, \quad (\text{B9})$$

where $h(x) := \frac{H_N(x)}{\sqrt{2^N N!}}$ with $H_N(x)$ a Hermite polynomial.

Note that indeed $\int_{\mathbb{R}} \tilde{Q}_{q|\theta} dq = 1$ and $\tilde{Q}_{q|\theta} \geq 0$, so this is a valid qubit POVM with continuous outcomes. We then have $A_1 = \tilde{Q}_0$ and $A_2 = \tilde{Q}_\theta$, and this pair is incompatible (despite the truncation), if $\theta \neq 0, \pi$. Indeed, since these are rank-1 POVMs, they can only be compatible if $\tilde{Q}_{q|\theta} = \int D(q|q') \tilde{Q}_{q|0}$ for some classical postprocessing $D(q|q')$ [51] implying $e^{i\theta} \in \mathbb{R}$, a contradiction. Hence the pure NOON-state (no damping, $r = 1$) is steerable with these measurements. With $0 < r < 1$, the “damped” observables become

$$\mathsf{T}_r^*(\tilde{Q}_{q|\theta}) = \begin{pmatrix} 1 & r e^{-iN\theta}h(q) \\ r e^{iN\theta}h(q) & r^2 h(q)^2 + 1 - r^2 \end{pmatrix} \frac{e^{-q^2}}{\sqrt{\pi}}. \quad (\text{B10})$$

The determinant of this kernel matrix is $(1-r^2)\frac{e^{-2q^2}}{\pi}$ so that the compatibility criterion in Eq. (B1) reads $r^2 \leq 1/2$, hence, $r_c \geq 1/\sqrt{2}$, corresponding to $\eta_c \geq 2/3$ (independently of q and N). The value $\eta_c \approx 2/3$ has previously been obtained numerically [19]; up to our knowledge, ours is the first fully analytical proof of a lower bound on η_c . Note that when $\eta \leq 2/3$, Eq. (B3) gives an

N	4	6	8	10	12	14	16	18	20
η	0.742	0.698	0.684	0.678	0.675	0.674	0.673	0.672	0.671

TABLE I. Minimal η such that the obtained N -valued observables become incompatible.

explicit joint observable and hence a local hidden state model for the two quadrature measurements on the state at hand.

In order to prove an analytic upper bound for r_c , we binarise the POVMs \tilde{Q}_θ . Incompatibility of binarisations is sufficient for that of the original POVMs, as coarse-graining is an instance of postprocessing. Choosing the split at $q = 0$ (i.e. Alice only records if $q > 0$ or not) gives the POVM with elements $\frac{1}{2}(\mathbb{1} \pm \mathbf{n} \cdot \boldsymbol{\sigma})$ where $\mathbf{n} = (2r\sqrt{2}/\pi)(\cos \theta, \sin \theta, 0)$. Using an exact criterion [42] we conclude that the binarisations are incompatible for $r^2 \geq \pi(1 - \sin \theta)/(2 \cos^2 \theta)$. Notice that the bound depends on θ ; with $\theta = \pi/2$ (orthogonal quadratures) we get $r_c \leq \sqrt{\pi}/2$, or $\eta_c \leq 2\pi/(4 + \pi)$. Since the split at $q = 0$ is the most incompatible binarisation of quadratures [52], finer coarse-grainings are needed to get better bounds, see below.

3. Coarse-graining and numerical methods

In this section, we will discuss the numerical methods used to derive upper bounds on η_c , namely, $\eta_c \geq 2/3$ for all N , and $\eta_c \leq 0.671$ for $N = 1$.

For this we use semidefinite programming. More precisely, we use the incompatibility robustness (IR) defined in Eq. (A15), and see for which values of η_c we have $\text{IR} > 0$. We split \mathbb{R} into the intervals $(-\infty, -c]$, $[-c, -c + c/N_{\text{int}}]$, \dots , $[-c/N_{\text{int}}, 0]$, \dots , $[0, c/N_{\text{int}}]$, \dots , $[c, \infty)$, where $c \approx 1.4$. The corresponding qubit observables are obtained by integrating over the intervals I_k , i.e., $Q_{I_k|\theta} = \int_{I_k} \tilde{Q}_{q|\theta} dq$, and such integrals can be explicitly written in terms of error functions. The results are summarized in Tab. I. With increasing precision we get quite close to the bound $2/3$ for $\theta = \pi/2$ – with 20 intervals we have 0.671.

The same procedure can be applied for the same choice of intervals but different choices of angles, in particular for pairs $(0, \theta)$, as shown in Fig. 1 in the main text.

One can try the same approach in the different subspaces with a higher number of photons. For instance, we investigated the case of 0 or 6 photons, which turned out to be more sensitive to noise, e.g., for the case of $N_{\text{int}} = 16$ one can reach $\eta_{\text{min}} = 0.89$. If one further increases the number of intervals, the computation becomes too slow and practically impossible.

Appendix C: Derivation of the Gaussian steering criterion in the channel picture

In this section, we first review the basic notions and definitions related to Gaussian steering. Then, we will prove Lemma 3 and Prop. 4.

1. Characteristic functions of Gaussian states, measurements, and channels

For the convenience of the reader, we now review the characteristic function formalism for Gaussian quantum objects [43, 44]; see also [29, 45]. This representation treats all of them in the same footing, is a transparent quantum analogue of the corresponding classical objects by way of a rigorous correspondence theory [53], does not require the use of ancillas, circumvents the technical problem of the POVM elements not always being proper operators (see the discussion in the main text), and is especially convenient to use with concatenation, making explicit the idea that a Gaussian channel applied to a Gaussian state (Schrödinger picture) or measurement (Heisenberg picture) produces a new Gaussian state and measurement, respectively. We note that this approach differs from the alternative (equivalent) one introduced by Giedke and Cirac [54], on which Wiseman *et al.* based their derivation of the Gaussian steering criterion [1].

An optical system with N modes is a continuous variable (CV) quantum system (see e.g. [43]) with the infinite-dimensional Hilbert space $\mathcal{H}^{\otimes N} = \bigotimes_{j=1}^N L^2(\mathbb{R}) \simeq L^2(\mathbb{R}^N)$. The associated phase space is \mathbb{R}^{2N} , with canonical coordinates $\mathbf{x} = (q_1, p_1, \dots, q_N, p_N)^T$ in a fixed symplectic basis. The corresponding standard quadrature operators are denoted by Q_j and P_j ; they satisfy $[Q_i, Q_j] = i\delta_{ij}\mathbb{1}$, and we set $\mathbf{R} = (Q_1, P_1, \dots, Q_N, P_N)^T$, so that $[R_i, R_j] = i\Omega_{ij}\mathbb{1}$ with $\Omega = \bigoplus_{j=1}^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The Weyl operators $W(\mathbf{x}) = e^{-i\mathbf{x}^T \mathbf{R}}$ satisfy the canonical commutation relation (CCR)

$$W(\mathbf{x})W(\mathbf{y}) = e^{-i\mathbf{x}^T \Omega \mathbf{y}} W(\mathbf{y})W(\mathbf{x}), \quad (\text{C1})$$

and we define displacement operators $D_{\mathbf{c}} := W(\Omega^T \mathbf{c})$ so that $D_{\mathbf{c}}^* W(\mathbf{x}) D_{\mathbf{c}} = e^{-i\mathbf{c}^T \mathbf{x}} W(\mathbf{x})$. A matrix \mathbf{S} is *symplectic* if $\mathbf{S}^T \Omega \mathbf{S} = \Omega$; then by Stone-von Neumann theorem there is a unitary $U_{\mathbf{S}}$ with $U_{\mathbf{S}}^* W(\mathbf{x}) U_{\mathbf{S}} = W(\mathbf{S}\mathbf{x})$.

A state on a CV system is *Gaussian* if its characteristic function $\hat{\rho}(\mathbf{x}) := \text{tr}[\rho W(\mathbf{x})]$ is a Gaussian function:

$$\hat{\rho}(\mathbf{x}) = e^{-\frac{1}{4}\mathbf{x}^T \mathbf{V}_\rho \mathbf{x} - i\mathbf{r}^T \mathbf{x}} \quad (\text{C2})$$

where \mathbf{V}_ρ is the *covariance matrix* (CM) $[V_\rho]_{ij} = \text{tr}[\rho\{R_i - r_i, R_j - r_j\}]$ with displacement vector $r_j = \text{tr}[\rho R_j]$. The CM satisfies the *uncertainty relation*

$$\mathbf{V}_\rho + i\Omega \geq 0. \quad (\text{C3})$$

Crucially, *every* real and symmetric matrix \mathbf{V} satisfying (C3) is a CM of some Gaussian state ρ .

A measurement (POVM) $M_{\mathbf{a}}$ with outcomes $\mathbf{a} \in \mathbb{R}^d$, is *Gaussian* if its outcome distribution for any Gaussian state is a Gaussian (i.e. Normal) distribution. This is the case when the operator-valued characteristic function $\hat{M}(\mathbf{p}) := \int e^{i\mathbf{p}^T \mathbf{a}} M_{\mathbf{a}} d\mathbf{a}$ is of the form

$$\hat{M}(\mathbf{p}) = W(\mathbf{K}\mathbf{p}) e^{-\frac{1}{4}\mathbf{p}^T \mathbf{L}\mathbf{p} - i\mathbf{m}^T \mathbf{p}}, \quad (\text{C4})$$

where \mathbf{K} is an $N \times d$ -matrix and \mathbf{L} is an $d \times d$ -matrix satisfying the positivity condition

$$\mathbf{C}_{\mathbf{K},\mathbf{L}} := \mathbf{L} - i\mathbf{K}^T \Omega \mathbf{K} \geq 0, \quad (\text{C5})$$

and \mathbf{m} is a displacement vector. Importantly, *every* triple $(\mathbf{K}, \mathbf{L}, \mathbf{m})$ satisfying (C5) defines a Gaussian measurement. In the case $d = 1$ we have $\mathbf{K} = \mathbf{x}$, a column vector, while $\mathbf{L} = 2\xi^2$ and $\mathbf{m} = m$ are just numbers. Since shifts in outcomes are irrelevant for steering, we consider $m = 0$ so that the corresponding POVM $M_{a|\mathbf{x},\xi}$ has characteristic function $\hat{M}_{a|\mathbf{x},\xi}(p) = e^{-ip\mathbf{x}^T \mathbf{R}} e^{-\frac{1}{2}p^2 \xi^2}$, which implies that $M_{a|\mathbf{x},\xi}$ has the convolution form described in the main text. With $\xi^2 = 0$ we simply obtain the PVM with characteristic function $\hat{M}(p) = e^{-ip\mathbf{x}^T \mathbf{R}}$, that is, the unitary group generated by the quadrature $Q_{\mathbf{x}} = \mathbf{x}^T \mathbf{R}$. Hence Gaussian PVMs with $d = 1$ are just quadrature measurements, as stated in the main text.

A quantum channel between two CV systems with respective degrees of freedom N and N' is *Gaussian*, if it maps Gaussian states into Gaussian states. In the Heisenberg picture, this entails

$$\Lambda^*(W(\mathbf{x})) = W(\mathbf{M}\mathbf{x}) e^{-\frac{1}{4}\mathbf{x}^T \mathbf{N}\mathbf{x} - i\mathbf{c}^T \mathbf{x}} \quad (\text{C6})$$

where \mathbf{M} is a real $2N \times 2N'$ -matrix, and \mathbf{N} is a real $2N' \times 2N'$ -matrix. Due to complete positivity, they satisfy

$$\mathbf{C}_{\mathbf{M},\mathbf{N}} + i\Omega \geq 0, \quad (\text{C7})$$

where (interestingly) $\mathbf{C}_{\mathbf{M},\mathbf{N}}$ is as in (C5). Again, *every* triple $(\mathbf{M}, \mathbf{N}, \mathbf{c})$ with (C7) defines a Gaussian channel via (C6). Unitary channels $B \mapsto U^* B U$ have $\mathbf{N} = \mathbf{0}$ and $\mathbf{M} = \mathbf{S}$ symplectic, i.e. $U = D_{\mathbf{c}} U_{\mathbf{S}}$. Using (C2) and (C6) we get the general transformation rule for states in terms of CMs and displacement vectors:

$$\mathbf{V} \mapsto \mathbf{M}^T \mathbf{V} \mathbf{M} + \mathbf{N}, \quad \mathbf{r} \mapsto \mathbf{M}^T \mathbf{r} + \mathbf{c}. \quad (\text{C8})$$

Similarly, a Gaussian channel with matrices $(\mathbf{M}, \mathbf{N}, \mathbf{c})$, followed by a Gaussian measurement with matrices $(\mathbf{K}, \mathbf{L}, \mathbf{m})$ is clearly a Gaussian measurement as well, and we can easily derive the associated matrices by combining (C4) and (C6); there the result is

$$(\mathbf{K}, \mathbf{L}, \mathbf{m}) \mapsto (\mathbf{M}\mathbf{K}, \mathbf{L} + \mathbf{K}^T \mathbf{N} \mathbf{K}, \mathbf{m} + \mathbf{K}^T \mathbf{c}). \quad (\text{C9})$$

Using (C9) we observe that (for $\mathbf{c} = 0$) the channel transforms a quadrature PVM $Q_{\mathbf{x}}$ into the noisy POVM $M_{\mathbf{M}\mathbf{x},\xi}$ where now $\xi^2 = \mathbf{x}^T \mathbf{N} \mathbf{x} / 2$.

Finally, a Gaussian post-processing (*classical* channel) is one which transforms every Gaussian probability distribution into another one. These are determined

by triples $(\mathbf{M}, \mathbf{N}, \mathbf{c})$ as in the above quantum case, *except that only $\mathbf{N} \geq \mathbf{0}$ is required* as complete positivity does not appear in the classical case. One can show that the matrices are associated with linear coordinate transformations, convolutions, and translations, respectively [29]. Note that linear transformations include the deterministic post-processings which simply project on a lower-dimensional subspace. A Gaussian measurement $(\mathbf{K}, \mathbf{L}, \mathbf{m})$, followed by a Gaussian postprocessing $(\mathbf{M}, \mathbf{N}, \mathbf{c})$, is again a Gaussian measurement, with parameters obtained by the transformation rule

$$(\mathbf{K}, \mathbf{L}, \mathbf{m}) \mapsto (\mathbf{K}\mathbf{M}, \mathbf{N} + \mathbf{M}^T \mathbf{L} \mathbf{M}, \mathbf{c} + \mathbf{M}^T \mathbf{m}). \quad (\text{C10})$$

2. Proof of the Gaussian state-channel duality (Lemma 2)

We suppose Alice and Bob both have an identical continuous variable system with N degrees of freedom. In order to preserve Gaussianity, we need to do the diagonalisation of the reference state σ “symplectically” (see e.g. [55]): Let \mathbf{V}_{σ} be the CM of σ and \mathbf{r}_{σ} the displacement. By Williamson’s theorem [56] there is a symplectic matrix \mathbf{S} such that $\mathbf{V}_{\sigma} = \mathbf{S}^T \mathbf{D} \mathbf{S}$ with $\mathbf{D} = \bigoplus_{k=1}^N \nu_k \mathbb{1}_2$, where ν_k are the symplectic eigenvalues of \mathbf{V}_{σ} , and we assume $\nu_i > 1$ (full symplectic rank). This is not restrictive as any $\nu_i = 1$ corresponds to a vacuum mode, which we may factor out from the system. Then $U = D_{\mathbf{r}_{\sigma}} U_{\mathbf{S}}$ diagonalises σ in the photon number basis $|\mathbf{n}\rangle = |n_1, \dots, n_N\rangle$:

$$U^* \sigma U = \sum_{\mathbf{n}} p_{\mathbf{n}} |\mathbf{n}\rangle \langle \mathbf{n}|, \quad p_{\mathbf{n}} = \prod_{k=1}^N \frac{2}{1 + \nu_k} \left(\frac{\nu_k - 1}{\nu_k + 1} \right)^{n_k}. \quad (\text{C11})$$

Moreover, the purification $\sum_{\mathbf{n}} \sqrt{p_{\mathbf{n}}} |\mathbf{n}\rangle \otimes |\mathbf{n}\rangle$ has the CM

$$\begin{pmatrix} \mathbf{D} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{D} \end{pmatrix} \quad \text{with } \mathbf{Z} = \bigoplus_{i=1}^N \sqrt{\nu_i^2 - 1} \sigma_z. \quad (\text{C12})$$

The eigenbasis $\{U|\mathbf{n}\rangle\}$ of σ is the one we use to construct the steering channels following the general scheme (see Prop. 1). Hence we form the purification

$$\Omega_{\sigma} = \sum_{\mathbf{n}} \sqrt{p_{\mathbf{n}}} U|\mathbf{n}\rangle \otimes U|\mathbf{n}\rangle \quad (\text{C13})$$

which by (C8) has displacement vector $\mathbf{r}_{\sigma} \oplus \mathbf{r}_{\sigma}$ and CM

$$\mathbf{V}_{\Omega_{\sigma}} = (\mathbf{S}^T \oplus \mathbf{S}^T) \begin{pmatrix} \mathbf{D} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{D} \end{pmatrix} (\mathbf{S} \oplus \mathbf{S}) = \begin{pmatrix} \mathbf{V}_{\sigma} & \mathbf{S}^T \mathbf{Z} \mathbf{S} \\ \mathbf{S}^T \mathbf{Z} \mathbf{S} & \mathbf{V}_{\sigma} \end{pmatrix} \quad (\text{C14})$$

as stated in the main text. Again by (C8), the application of a Gaussian channel Λ with matrices $(\mathbf{M}, \mathbf{N}, \mathbf{c})$ yields

the state $\rho := (\Lambda \otimes \text{Id})(|\Omega_\sigma\rangle\langle\Omega_\sigma|)$ with CM

$$\begin{aligned} \mathbf{V} &= (\mathbf{M}^T \oplus \mathbf{I}) \begin{pmatrix} \mathbf{V}_\sigma & \mathbf{S}^T \mathbf{Z} \mathbf{S} \\ \mathbf{S}^T \mathbf{Z} \mathbf{S} & \mathbf{V}_\sigma \end{pmatrix} (\mathbf{M} \oplus \mathbf{I}) + \mathbf{N} \oplus \mathbf{0} \\ &= \begin{pmatrix} \mathbf{M}^T \mathbf{V}_\sigma \mathbf{M} + \mathbf{N} & \mathbf{M}^T \mathbf{S}^T \mathbf{Z} \mathbf{S} \\ \mathbf{S}^T \mathbf{Z} \mathbf{S} \mathbf{M} & \mathbf{V}_\sigma \end{pmatrix}, \end{aligned} \quad (\text{C15})$$

and displacement $\mathbf{r} = (\mathbf{M}^T \mathbf{r}_\sigma + \mathbf{c}) \oplus \mathbf{r}_\sigma$. Now $\mathbf{V}_\rho + i\mathbf{\Omega} \geq 0$ if and only if $\mathbf{C} \geq 0$ where \mathbf{C} is the Schur complement of the block $\mathbf{V}_\sigma + i\mathbf{\Omega}_B$ in $\mathbf{V}_\rho + i\mathbf{\Omega}$. But

$$\begin{aligned} \mathbf{C} &= \mathbf{M}^T \mathbf{V}_\sigma \mathbf{M} + \mathbf{N} + i\mathbf{\Omega}_A \\ &\quad - \mathbf{M}^T \mathbf{S}^T \mathbf{Z} \mathbf{S} (\mathbf{V}_\sigma + i\mathbf{\Omega}_B)^{-1} \mathbf{S}^T \mathbf{Z} \mathbf{S} \mathbf{M} \\ &= \mathbf{N} + i\mathbf{\Omega}_A + \mathbf{M}^T \mathbf{S}^T (\mathbf{D} - \mathbf{Z}(\mathbf{D} + i\mathbf{\Omega}_B)^{-1} \mathbf{Z}) \mathbf{S} \mathbf{M} \\ &= \mathbf{C}_{\mathbf{M}, \mathbf{N}} + i\mathbf{\Omega}_A, \end{aligned} \quad (\text{C16})$$

where we have used $\mathbf{D} - \mathbf{Z}(\mathbf{D} + i\mathbf{\Omega})^{-1} \mathbf{Z} = \mathbf{\Omega}$ which is straightforward to verify. This shows that $\mathbf{C}_{\mathbf{M}, \mathbf{N}} + i\mathbf{\Omega}_A \geq 0$ is *equivalent* to \mathbf{V}_ρ being a valid CM. Now for any given Gaussian state ρ with CM and displacement vector

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_A & \mathbf{\Gamma}^T \\ \mathbf{\Gamma} & \mathbf{V}_\sigma \end{pmatrix}, \quad \mathbf{r} = \mathbf{r}_A \oplus \mathbf{r}_\sigma, \quad (\text{C17})$$

we can define

$$(\mathbf{M}, \mathbf{N}, \mathbf{c}) = ((\mathbf{S}^T \mathbf{Z} \mathbf{S})^{-1} \mathbf{\Gamma}, \mathbf{V}_A - \mathbf{M}^T \mathbf{V}_\sigma \mathbf{M}, \mathbf{r}_A - \mathbf{M}^T \mathbf{r}_\sigma), \quad (\text{C18})$$

which then satisfies (C15), so that $\mathbf{C}_{\mathbf{M}, \mathbf{N}} + i\mathbf{\Omega}_A \geq 0$ by the above equivalence, showing that $(\mathbf{M}, \mathbf{N}, \mathbf{c})$ determines a Gaussian channel Λ with $\rho = (\Lambda \otimes \text{Id})(|\Omega_\sigma\rangle\langle\Omega_\sigma|)$.

3. Proof of Lemma 3 – joint measurability of convoluted quadratures

This lemma was critical for the characterisation of Gaussian steering. In order to prove it we let $r = \mathbf{x}^T \mathbf{\Omega} \mathbf{y}$, so that $[Q_{\mathbf{x}}, Q_{\mathbf{y}}] = ir\mathbb{1}$. If $r = 0$ then $Q_{\mathbf{x}}$ and $Q_{\mathbf{y}}$ commute and the claim is trivial since they stay jointly measurable after convolution. We suppose $r > 0$, and look at the scaled quadrature $Q_{\mathbf{y}}/r = \mathbf{y}^T \mathbf{R}/r = Q_{\mathbf{y}/r}$. By using the connection $Q_{\mathbf{y}} = \int a Q_{a|\mathbf{y}} da$ between the operator $Q_{\mathbf{y}}$ and the corresponding PVM $Q_{a|\mathbf{y}}$, we see that $Q_{a|\mathbf{y}/r} = r Q_{ra|\mathbf{y}}$. A direct computation then shows that scaling of the noisy POVM gives $M_{a|\mathbf{y}/r, \xi'/r} = r M_{ra|\mathbf{y}, \xi'}$. Since scaling is a post-processing and hence does not affect joint measurability, the original pair $(M_{\mathbf{x}, \xi}, M_{\mathbf{y}, \xi'})$ is jointly measurable if and only if $(M_{\mathbf{x}, \xi}, M_{\mathbf{y}/r, \xi'/r})$ is. But the corresponding quadrature pair $(Q_{\mathbf{x}}, Q_{\mathbf{y}/r})$ is canonical, as $[Q_{\mathbf{x}}, Q_{\mathbf{y}/r}] = [\mathbf{x}^T \mathbf{R}, \mathbf{y}^T \mathbf{R}/r] = i(\mathbf{x}^T \mathbf{\Omega} \mathbf{y}/r)\mathbb{1} = i\mathbb{1}$, and hence unitarily equivalent to the pair $(Q_0, Q_{\pi/2})$ via a symplectic transformation, where $Q_\theta = (e^{i\theta} a^\dagger + e^{-i\theta} a)/\sqrt{2}$ are the rotated quadratures of a single-mode system. The same unitary then transforms the convoluted pair

$(M_{\mathbf{x}, \xi}, M_{\mathbf{y}/r, \xi'/r})$ into the pair $(M_{0, \xi}, M_{\pi/2, \xi'/r})$ where $M_{a|\theta, \xi} := \frac{1}{\sqrt{2\pi\xi}} \int e^{-\frac{1}{2}(a-a')^2/\xi^2} Q_{a'|\theta} da'$, and hence it suffices to show that the joint measurability of $(M_{0, \xi}, M_{\pi/2, \xi'/r})$ is equivalent to the inequality $\xi(\xi'/r) \geq 1/2$, and that the joint observable, when exists, can be chosen Gaussian.

In order to prove this, we use known results on joint measurability of “unsharp” position and momentum [46, 47], which is exactly what our convoluted quadratures are. In particular, if $(M_{0, \xi}, M_{\pi/2, \xi'/r})$ are jointly measurable, they must have a joint observable of the Weyl-covariant form $G_{a_1, a_2} = W(a_1, a_2) \rho_0 W(a_1, a_2)^*/(2\pi)$, where ρ_0 is a state with

$$\text{tr}[\rho_0 Q_{a_1|0}] = \frac{e^{-\frac{1}{2}a_1^2/\xi^2}}{\sqrt{2\pi\xi}}, \quad \text{tr}[\rho_0 Q_{a_2|\pi/2}] = \frac{e^{-\frac{1}{2}a_2^2/(\xi'/r)^2}}{\sqrt{2\pi(\xi'/r)}}. \quad (\text{C19})$$

This implies that ξ and ξ'/r are the standard deviations of Q_0 and $Q_{\pi/2}$ in the state ρ_0 , hence satisfying $\xi(\xi'/r) \geq 1/2$ by the Heisenberg uncertainty principle. Conversely, if the inequality holds, we can define $\rho_0 = |\psi_0\rangle\langle\psi_0|$ in the coordinate representation as $\psi_0(a) = (2c/\pi)^{\frac{1}{4}} e^{-(c+iw)a^2}$ with $\xi^2 = 1/(4c)$ and $\xi'^2/r^2 = (c^2 + d^2)/d$; then a direct computation shows that the corresponding G_{a_1, a_2} is a joint observable for $M_{0, \xi}$ and $M_{\pi/2, \xi'/r}$. This observable is Gaussian since ρ_0 is a Gaussian state [45].

Finally, since all the above unitary equivalences were done via symplectic transformations, the original POVMs have a Gaussian joint observable as well. This completes the proof.

4. Proof of Prop. 4 and the construction of quadrature pairs for Gaussian steering

This proposition contains our main results on Gaussian steering. In order to prove it, we first note that (ii) trivially implies (i). Next, we repeat the calculation (C16) without $\mathbf{\Omega}_A$, establishing that $\mathbf{C}_{\mathbf{M}, \mathbf{N}}$ is the Schur complement of $\mathbf{V}_\sigma + i\mathbf{\Omega}_B$ in $\mathbf{V}_\rho + i(\mathbf{0} \oplus \mathbf{\Omega}_B)$; this shows that (iii) and (iv) are equivalent. Furthermore, using [29, Prop. 2] we conclude that \mathbf{T}_ρ maps the set of all Gaussian measurements into a set having a joint (Gaussian) measurement, if $\mathbf{C}_{\mathbf{M}, \mathbf{N}} \geq 0$. Hence (i) implies (iv).

We are left with the proof of the main result, stating that (iv) implies (ii). Assuming (iv) let \mathbf{x}, \mathbf{y} be vectors such that $(\mathbf{y}^T - i\mathbf{x}^T) \mathbf{C}_{\mathbf{A}, \mathbf{B}} (\mathbf{y} + i\mathbf{x}) < 0$. Then by complete positivity $(\mathbf{y}^T - i\mathbf{x}^T) (\mathbf{C}_{\mathbf{M}, \mathbf{N}} + i\mathbf{\Omega}) (\mathbf{y} + i\mathbf{x}) \geq 0$, which implies $r := \mathbf{x}^T \mathbf{\Omega} \mathbf{y} > 0$ and

$$(\mathbf{M}\mathbf{x})^T \mathbf{\Omega} \mathbf{M}\mathbf{y} > \frac{1}{2}(\mathbf{x}^T \mathbf{N} \mathbf{x} + \mathbf{y}^T \mathbf{N} \mathbf{y}). \quad (\text{C20})$$

Clearly, we may replace \mathbf{x} and \mathbf{y} with $r^{-\frac{1}{2}}\mathbf{x}$ and $r^{-\frac{1}{2}}\mathbf{y}$ and (C20) still holds. Then the pair $Q_{\mathbf{x}} = \mathbf{x}^T \mathbf{R}$ and $P_{\mathbf{y}} = \mathbf{y}^T \mathbf{R}$ of quadratures is canonical since $\mathbf{x}^T \mathbf{\Omega} \mathbf{y} = 1$. It is easy to check using the transformation rule (C9) that the channel \mathbf{T}_ρ , having parameters $(\mathbf{M}, \mathbf{N}, \mathbf{c})$, transforms

the associated PVMs into POVMs $M_{\mathbf{M}\mathbf{x},\xi}$ and $M_{\mathbf{M}\mathbf{y},\xi'}$ (up to irrelevant shifts in outcomes) where $\xi^2 = \mathbf{x}^T \mathbf{N} \mathbf{x} / 2$ and $\xi'^2 = \mathbf{y}^T \mathbf{N} \mathbf{y} / 2$. By (C20) we have $2\xi\xi' \leq \xi^2 + \xi'^2 < (\mathbf{M}\mathbf{x})^T \mathbf{N} \mathbf{M}\mathbf{y}$ so from Lemma 3 we conclude that the POVMs are not jointly measurable. This means we have found a canonical pair $(Q_{\mathbf{x}}, Q_{\mathbf{y}})$ of quadratures such that $T_\rho(Q_{\mathbf{x}})$ and $T_\rho(Q_{\mathbf{y}})$ are not jointly measurable, so according to Prop. 1, the state ρ is steerable by this pair. Hence, (ii) holds.

This completes the proof, and also shows explicitly how one can construct quadrature pairs for which steering is possible when the conditions of the proposition hold.

5. The joint Gaussian measurement and the corresponding LHS in the non-steerable case

Here we explain the condition (iv) and the construction of the joint Gaussian measurement G_λ , $\lambda \in \mathbb{R}^{2N}$ in the non-steerable case. For the sake of completeness, we also show how one can easily derive the known Gaussian LHS model σ_λ for this case [1] from our formalism.

Following an argument in [29, Prop. 2], we first note that an arbitrary Gaussian measurement $(\mathbf{K}, \mathbf{L}, \mathbf{m})$ on Alice's side is transformed by the channel $(\mathbf{M}, \mathbf{N}, \mathbf{c})$ into one with parameters $(\mathbf{K}', \mathbf{L}', \mathbf{m}') = (\mathbf{M}\mathbf{K}, \mathbf{L} + \mathbf{K}^T \mathbf{N} \mathbf{K}, \mathbf{m} + \mathbf{K}^T \mathbf{c})$ by (C9). In order to show that such POVMs are all jointly measurable, we only need to *reinterpret the channel parameters $(\mathbf{M}, \mathbf{N}, \mathbf{c})$ as the joint measurement G_λ* . Indeed, with $(\mathbf{K}, \mathbf{L}, \mathbf{m})$ taken as post-processing parameters, (C10) becomes identical to (C9), showing how $(\mathbf{K}', \mathbf{L}', \mathbf{m}')$ is postprocessed from G_λ . We stress that the nontrivial part is in the positivity requirements, which are *not* in general identical. Indeed, the reinterpretation is possible only when (iv) does not hold, i.e. $\mathbf{C}_{\mathbf{M}, \mathbf{N}} \geq \mathbf{0}$, which is not true for general channels.

Finally, we show that our G_λ is consistent with the LHS of [1]. By Appendix A1, the joint POVM G_λ and the LHS σ_λ are related by $\sigma_\lambda = \sigma^{\frac{1}{2}} G_\lambda \sigma^{\frac{1}{2}} = \text{tr}_A[G_\lambda \otimes \mathbb{1}|\Omega_\sigma\rangle\langle\Omega_\sigma|]$. Now σ_λ has finite trace, and

$\tilde{\sigma}_\lambda := \sigma_\lambda / \text{tr}[\sigma_\lambda]$ is an actual state; we show that it is Gaussian by computing the characteristic function $\widehat{\tilde{\sigma}_\lambda}(\mathbf{x}) := \text{tr}[W(\mathbf{x})\tilde{\sigma}_\lambda] = f_\mathbf{x}(\lambda)/f_0(\lambda)$, where $f_\mathbf{x}(\lambda) := \text{tr}[W(\mathbf{x})\sigma_\lambda] = \text{tr}[G_\lambda \otimes W(\mathbf{x})|\Omega_\sigma\rangle\langle\Omega_\sigma|]$. For simplicity we assume $\mathbf{c} = 0$. Due to (C4), the function $f_\mathbf{x}$ is determined via its Fourier transform, in terms of (\mathbf{M}, \mathbf{N}) :

$$\begin{aligned} \widehat{f}_\mathbf{x}(\mathbf{p}) &= \int e^{i\mathbf{p}^T \lambda} \text{tr}[G_\lambda \otimes W(\mathbf{x})|\Omega_\sigma\rangle\langle\Omega_\sigma|] d\lambda \\ &= \text{tr}[\hat{G}(\mathbf{p}) \otimes W(\mathbf{x})|\Omega_\sigma\rangle\langle\Omega_\sigma|] \\ &= \text{tr}[W(\mathbf{M}\mathbf{p}) \otimes W(\mathbf{x})|\Omega_\sigma\rangle\langle\Omega_\sigma|] e^{-\frac{1}{4}\mathbf{p}^T \mathbf{N} \mathbf{p}}. \end{aligned} \quad (\text{C21})$$

Now by definition, the first factor in the last expression is the characteristic function of the state Ω_σ , evaluated at $\mathbf{M}\mathbf{p} \oplus \mathbf{x}$; hence by (C2) and (C15) we get

$$\begin{aligned} \widehat{f}_\mathbf{x}(\mathbf{p}) &= e^{-\frac{1}{4}((\mathbf{M}\mathbf{p})^T \oplus \mathbf{x}^T) \mathbf{V}_{\Omega_\sigma} (\mathbf{M}\mathbf{p} \oplus \mathbf{x})} e^{-\frac{1}{4}\mathbf{p}^T \mathbf{N} \mathbf{p}} \\ &= e^{-\frac{1}{4}(\mathbf{p}^T \oplus \mathbf{x}^T) \mathbf{V}(\mathbf{p} \oplus \mathbf{x})} = e^{-\frac{1}{4}(\mathbf{p}^T \mathbf{V}_A \mathbf{p} + 2\mathbf{p}^T \mathbf{\Gamma}^T \mathbf{x} + \mathbf{x}^T \mathbf{V}_\sigma \mathbf{x})} \\ &= e^{-\frac{1}{4}(\mathbf{p} - \boldsymbol{\mu}_\mathbf{x})^T \mathbf{V}_A (\mathbf{p} - \boldsymbol{\mu}_\mathbf{x})} e^{-\frac{1}{4}\mathbf{x}^T (\mathbf{V}_\sigma - \mathbf{\Gamma} \mathbf{V}_A^{-1} \mathbf{\Gamma}^T) \mathbf{x}} \end{aligned} \quad (\text{C22})$$

where $\boldsymbol{\mu}_\mathbf{x} = -\mathbf{V}_A^{-1} \mathbf{\Gamma}^T \mathbf{x}$, and we have used the notation (C17). Taking the inverse Fourier transform we obtain

$$f_\mathbf{x}(\lambda) = C e^{-\lambda^T \mathbf{V}_A^{-1} \lambda - i\lambda^T \boldsymbol{\mu}_\mathbf{x}} e^{-\frac{1}{4}\mathbf{x}^T (\mathbf{V}_\sigma - \mathbf{\Gamma} \mathbf{V}_A^{-1} \mathbf{\Gamma}^T) \mathbf{x}} \quad (\text{C23})$$

where C depends only on \mathbf{V}_A . Hence $\widehat{\tilde{\sigma}_\lambda}(\mathbf{x}) = f_\mathbf{x}(\lambda)/f_0(\lambda) = e^{-\frac{1}{4}\mathbf{x}^T (\mathbf{V}_\sigma - \mathbf{\Gamma} \mathbf{V}_A^{-1} \mathbf{\Gamma}^T) \mathbf{x} + i(\mathbf{\Gamma} \mathbf{V}_A^{-1} \lambda)^T \mathbf{x}}$, so by (C2), $\tilde{\sigma}_\lambda$ is Gaussian with CM and displacement

$$\mathbf{V}_\lambda = \mathbf{V}_\sigma - \mathbf{\Gamma} \mathbf{V}_A^{-1} \mathbf{\Gamma}^T, \quad \mathbf{r}_\lambda = -\mathbf{\Gamma} \mathbf{V}_A^{-1} \lambda. \quad (\text{C24})$$

Furthermore, each $\tilde{\sigma}_\lambda$ occurs in the LHS decomposition with Gaussian probability $p_\lambda = \text{tr}[\sigma_\lambda] = f_0(\lambda) \propto e^{-\lambda^T \mathbf{V}_A^{-1} \lambda}$. By changing the hidden variable λ to \mathbf{r}_λ we recover exactly the LHS of [1].

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